3001. We differentiate the area, using the product rule:

$$\begin{aligned} A &= \tfrac{1}{2}h(a+b) \\ \implies \tfrac{dA}{dt} &= \tfrac{1}{2}\tfrac{dh}{dt}(a+b) + \tfrac{1}{2}h\left(\tfrac{da}{dt} + \tfrac{db}{dt}\right) \\ &= \tfrac{1}{2}\cdot 6(2+3) + \tfrac{1}{2}\cdot 4(1+2) \\ &= 21. \end{aligned}$$

So, the rate is 21 square units per second.

3002. We rewrite the improper fraction as

$$\frac{2x+3}{x-1} \equiv \frac{2(x-1)+5}{x-1} \equiv 2 + \frac{5}{x-1}.$$

This gives

$$\int 2 + \frac{5}{x-1}\,dx = 2x + 5\ln|x-1| + c.$$

3003. The derivatives are $\frac{dy}{dx} = 2x$ and $\frac{dy}{dx} = -\frac{1}{2}x$. So, for perpendicularity, we need

$$2x \cdot -\tfrac{1}{2}x = -1 \implies x = \pm 1.$$

From the first parabola, the intersections are at $(\pm 1, 1)$. Substituting $(1,1)$ into the second parabola, $1 = \frac{1}{4} + k$, so $k = \frac{5}{4}$.

3004. Assume, for a contradiction, that three forces act on an object as in the question, and that the object is in equilibrium.

In the y direction, $1+2=3$, so the 3 N force must oppose the other two. Hence, since the object is not rotating, the 3 N force must lie between the other two, with line of action $x=2$. But, taking moments about $x=2$, the other two forces have total moment $2\times 1 - 1\times 1 \neq 0$. So, the object rotates. This is a contradiction.

Hence, the object cannot remain in equilibrium.

3005. The derivative of $\cot x$ is $-\operatorname{cosec}^2 x$. So, using the chain rule,

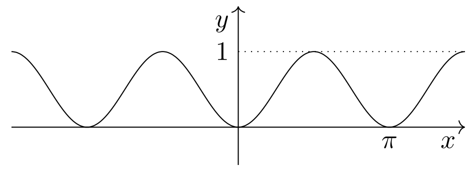
$$\tfrac{dy}{dx} = -3\operatorname{cosec}^2 3x.$$

Substituting in, we simplify the LHS with the third Pythagorean trig identity:

$$\begin{aligned} &\tfrac{dy}{dx} + 3 + 3y^2 \\ =\ &-3\operatorname{cosec}^2 3x + 3 + 3\cot^2 3x \\ \equiv\ &-3+3 \\ =\ &0. \end{aligned}$$

So, if $y = \cot 3x$, then $\frac{dy}{dx} + 3 + 3y^2 = 0$.

3006. Squaring both sides, $y = \sin^2 x$. Using a double-angle identity, this is $y = \frac{1}{2}(1-\cos 2x)$, which can be considered as a transformed cosine graph. It has $y \geq 0$, so every point on it also satisfies the original equation. Hence, the curve is as follows:



3007. (a) On the 26th day, the probability that the farmer starts is $50\% + 25\times 2\% = 100\%$. Prior to this there is a chance that he hasn't started. So, the answer is 26th August.

(b) The probability that the farmer *hasn't* started by the end of August nth is

$$\begin{aligned} &\frac{50}{100}\times\frac{48}{100}\times ... \times\frac{50-2(n-1)}{100} \\ \equiv\ &\frac{25}{50}\times\frac{24}{50}\times ... \times\frac{25-n+1}{50} \\ \equiv\ &\frac{25\times 24\times ... \times(25-n+1)}{50^n}. \end{aligned}$$

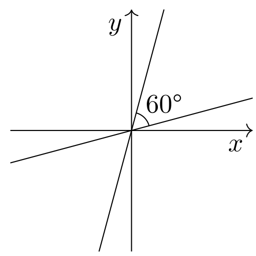
Multiplying top and bottom of the fraction by $(25-n)!$ completes the factorial 25! on the top, giving the probability he hasn't started as

$$\frac{25!}{50^n(25-n!)}.$$

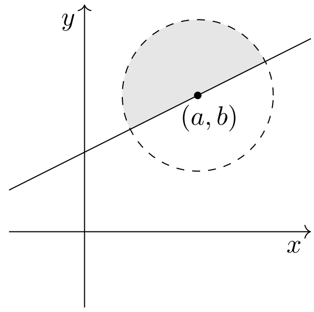
Therefore, the probability that the farmer *has* started by the end of August nth is

$$p_n = 1 - \frac{25!}{50^n(25-n!)}, \text{ as required.}$$

3008. The angle of inclination of gradient m is given by $\tan\theta = m$. So, the angles of inclination of the two lines in question are $\arctan\left(2+\sqrt{3}\right) = 75°$ and $\arctan\left(2-\sqrt{3}\right) = 15°$. The difference between these is 60°, as required.



3009. The first boundary equation is a straight line with gradient $\frac{1}{2}$, through the point (a,b). The second boundary equation is a circle, radius 1, centred on (a,b). So, the region is as follows:

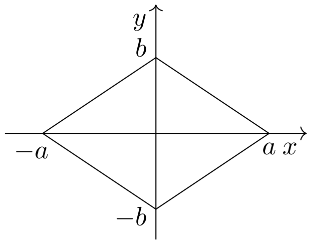


3010. (a) This is true. The combined mean is a weighted average of the two individual means.

(b) This is false. If the two samples each have low standard deviation, yet the difference in means is large, then the combined sample can have a much larger standard deviation than either of the individual ones.

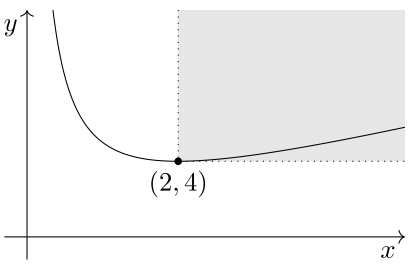
(c) This is true. If $\bar{x}_1 = \bar{x}_2$, then s_1, s_2 and s all measure deviation from $\bar{x}_1 = \bar{x}_2 = \bar{x}$. So, s is a weighted average (squared, summed, square-rooted) of the individual standard deviations s_1 and s_2, and must lie between them.

3011. In the first quadrant, x and y are positive, so the equation is $\frac{x}{a} + \frac{y}{b} = 1$. This is a line segment, with endpoints $(a, 0)$ and $(0, b)$. Since both x and y have mod functions applied to them, this line segment is reflected in the x and y axes, to produce 4 line segments forming a quadrilateral with vertices at $(\pm a, 0)$, $(0, \pm b)$.



By symmetry, this quadrilateral is a rhombus.

3012. The graph $y = x + \frac{4}{x}$ has asymptotes at $x = 0$ and $y = x$. For SPs, $1 - 4x^{-2} = 0$, so $x = \pm 2$. So, in the positive quadrant, the curve is as follows:



The least values of a and b occur for the largest possible domain and codomain, stretching to ∞, over which the function is invertible. This is to the right of and above the turning point, as shown by the shaded area. So, $a = 2$ and $b = 4$.

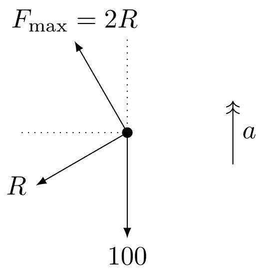
3013. (a) A, B, C have the same z coordinate, so we need only consider (x, y). Squared distances are

$$|AB|^2 = \left(\sqrt{8/9} + \sqrt{2/9}\right)^2 + 2/3 = 8/3,$$
$$|BC|^2 = \left(2\sqrt{2/3}\right)^2 = 8/3.$$

And, by symmetry, $|AB| = |AC|$. So, triangle ABC is equilateral, with side length $8/3$.

(b) Since points A, B, C all have z coordinate $-1/3$, we calculate the squared distances from each to $(0, 0, -1/3)$ on the z axis. These are each $8/9$. So, the squared distance between D and each other point is $8/9 + (4/3)^2 = 8/3$. Hence, $ABCD$ is a regular tetrahedron, side length $8/3$.

3014. (a) The part is symmetrical, so horizontal and rotational equilibrium are both guaranteed. Modelling the right-hand half as a particle, the force diagram, assuming limiting friction, is



From the lengths of the trapezium, the sides of the trapezium are sloped away from vertical at

$$\arcsin \frac{4.5 - 2.5}{4} = 30°.$$

i. Vertical equilibrium gives

$$2R \cos 30° - R \sin 30° - 100 = 0.$$

So $R = 81.2$ N. The total contact force is

$$C = \sqrt{(2R)^2 + R^2} = 181 \text{ N (3sf)}.$$

ii. The vertical equation of motion is

$$2R \cos 30° - R \sin 30° - 100 = \frac{100}{g} \times 0.5.$$

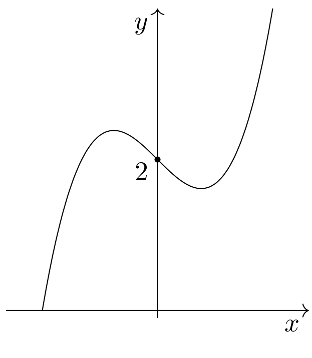
So $R = 85.3$ N. The total contact force is

$$C = \sqrt{(2R)^2 + R^2} = 191 \text{ N (3sf)}.$$

3015. Consider $y = x^3 - x$. Substituting $(-x)$ for x,

$$y = (-x)^3 - (-x) \equiv -(x^3 - x).$$

Hence, the curve $y = x^3 - x$ has odd symmetry, i.e. rotational symmetry about the origin. Translating this curve by vector $2\mathbf{j}$, $y = x^3 - x + 2$ therefore has rotational symmetry around $(0, 2)$. □



——— Alternative Method ———

The derivative is $\frac{dy}{dx} = 3x^2 + 1$. This is symmetrical in $x = 0$, so the gradients are identical for positive and negative x. Hence, the curve has rotational symmetry about its y intercept. This is $(0, 2)$. □

3016. Consider the cases in which f is sin or cos:

- f is sin: over the domain $[0, \pi/4)$, $\sin(x) > \frac{1}{2}$ if and only if $x > \frac{\pi}{6}$. And
$$\mathbb{P}\left(x > \tfrac{\pi}{6}\right) = 1 - \frac{\frac{\pi}{6}}{\frac{\pi}{4}} = \tfrac{1}{3}.$$
- f is cos: over the domain $[0, \pi/4)$, $\cos(x) > \frac{1}{2}$ is guaranteed.

So, the probability is $\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times 1 = \frac{2}{3}$.

3017. Substituting, we have $6 = A$ and then $3 = 6e^{-k}$, so $k = \ln 2$.

The velocity is positive for all t, so the distance is equal to the displacement. This is given by
$$\begin{aligned} d &= \int_0^1 6e^{-t\ln 2}\,dt \\ &= \left[-\tfrac{6}{\ln 2}e^{-t\ln 2}\right]_0^1 \\ &= \left(-\tfrac{3}{\ln 2}\right) - \left(-\tfrac{6}{\ln 2}\right) \\ &= \tfrac{3}{\ln 2}. \end{aligned}$$
The distance is $\frac{3}{\ln 2}$ metres.

3018. Taking out a factor of x,
$$\begin{aligned} &xy^2 - \frac{2x}{y} = k \\ \equiv\ & x\left(y^2 - 2y^{-1}\right) = k. \end{aligned}$$
Now, we are looking for tangents parallel to the y axis (perpendicular to the line $y = -1$), so we differentiate implicitly with respect to y:
$$\begin{aligned} & x\left(y^2 - 2y^{-1}\right) = k \\ \Longrightarrow\ & \tfrac{dx}{dy}\left(y^2 - 2y^{-1}\right) + x\left(2y + 2y^{-2}\right) = 0. \end{aligned}$$
Substituting $y = -1$, the RH bracket is zero and the LH bracket is not, which gives $\frac{dx}{dy} = 0$. Hence, at $y = -1$, the tangent is parallel to the y axis. Therefore, the curve intersects $y = -1$ at right angles.

3019. Setting the output to y, we rearrange:
$$\begin{aligned} & y = \frac{ax + b}{cx + d} \\ \Longrightarrow\ & cxy + dy = ax + b \\ \Longrightarrow\ & cxy - ax = b - dy \\ \Longrightarrow\ & x = \frac{b - dy}{cy - a}. \end{aligned}$$
So, $f^{-1}(x) = \dfrac{b - dx}{cx - a}$.

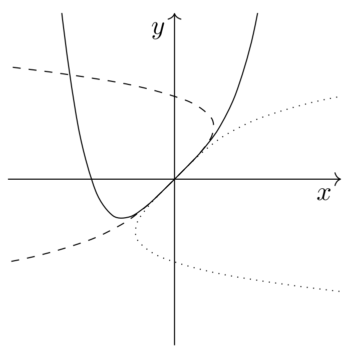
3020. Let the cuboid have two faces of area A_1 and A_2. Shifting the orientation from one to the other, the area in frictional contact has been scaled by $\frac{A_2}{A_1}$.

But the reaction acting on the cuboid is the same, so the pressure has been scaled inversely, by $\frac{A_1}{A_2}$.

So, while the area in contact scales up, the local pressure scales down. The net effect is no change on the frictional force.

3021. Let $u = 2 - \sqrt{x}$. Then
$$\begin{aligned} & \frac{du}{dx} = -\tfrac{1}{2}x^{-\frac{1}{2}} \\ \Longrightarrow\ & dx = (2u - 4)\,du. \end{aligned}$$
Also, the numerator is $2 + \sqrt{x} = 4 - u$. The new limits are $u = 2$ to $u = 1$. We can now enact the substitution:
$$\begin{aligned} & \int_0^1 \frac{2 + \sqrt{x}}{2 - \sqrt{x}}\,dx \\ =\ & \int_2^1 \frac{4 - u}{u}(2u - 4)\,du. \end{aligned}$$
Multiplying out, this is
$$\begin{aligned} & \int_2^1 12 - 2u - \tfrac{16}{u}\,du \\ =\ & \Big[12u - u^2 - 16\ln|u|\Big]_2^1 \\ =\ & (11 - 16\ln 1) - (20 - 16\ln 2) \\ =\ & 16\ln 2 - 9, \text{ as required.} \end{aligned}$$

3022. Reflection in $y = -x$ is equivalent to a reflection in $y = x$ (giving the dotted curve in the example below), followed by a rotation by 180° around the origin (giving the dashed curve).



Performing the above algebraically:

To reflect in $y = x$, we switch x and y, giving $x = f(y)$. Then rotation 180° around the origin is equivalent to reflection in both the x and y axes. To enact this, we replace y by $-y$ and x by $-x$. This gives $-x = f(-y)$.

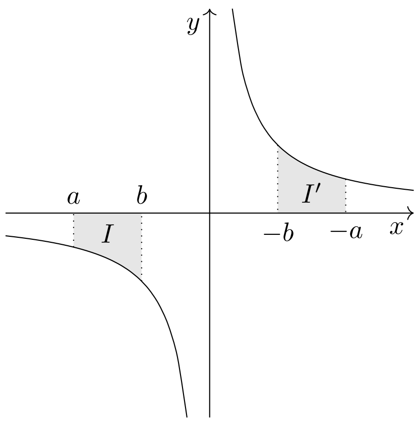
3023. We can consider the numbers as OOOEEE. There are ${}^6\text{C}_3 = 20$ orders of these letters, of which two, OEOEOE and EOEOEO, alternate even/odd. So, the probability is $\frac{1}{10}$.

— Alternative Method —

Place 1 without loss of generality. Then fill an adjacent spot. The probability that this number is even is $3/5$. Then the probability that the next spot is odd is $2/4$, and so on. This gives

$$1 \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{10}.$$

3024. Consider the graph of $y = 1/x$, which has rotational symmetry around the origin.



The integral of $1/x$ between $x = a$ and $x = b$ should calculate the signed area I, which is a negative quantity. It is the negative of I', which we know is given by $\ln(-a) - \ln(-b)$. Hence, we have

$$I = \ln(-b) - \ln(-a) = \Big[\ln(-x)\Big]_a^b.$$

— Nota Bene —

The result above is what justifies the use of the modulus function in $\int 1/x \, dx = \ln|x| + c$.

3025. Taking logs base x of both sides of $y = x^k$,

$$\log_x y = 2 + \sqrt{3}.$$

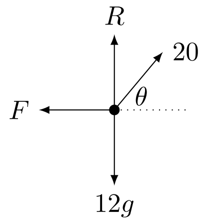
Switching base and input reciprocates the value of the logarithm, so

$$\log_y x = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}.$$

Adding the logarithms, the surds cancel. So, (x, y) points on the curve satisfy $\log_x y + \log_y x = 4$.

3026. The string can be horizontal, at $\theta = 0$. This is the (attainable) lower bound.

At the other extreme, the boundary case occurs in one of two ways: ① when the sledge is just about to lift off the ground, or ② when the sledge is moving but not accelerating. The force diagram, modelling the sledge as a particle, is



① If the sledge is about to lift, then the reaction is zero. Hence, $20\sin\theta = 12g$. But $\frac{12g}{20} > 1$, so no values of θ satisfy it. Hence, the sledge stays on the ground.

② If the sledge is moving but not accelerating, then friction is at F_{max}:

$$\begin{aligned} \updownarrow &: 20\sin\theta + R - 12g = 0, \\ \leftrightarrow &: 20\cos\theta - 0.125R = 0. \end{aligned}$$

Eliminating R, we get $\theta = 50.3°$ (3sf).

The upper limit is not attainable, because we are told that the sledge is speeding up. So, to three significant figures, $\theta \in [0, 50.3°)$.

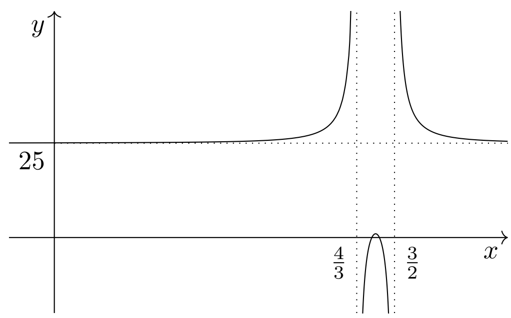
3027. (a) By the quotient rule,

$$\text{f}'(x) = \frac{17 - 12x}{(2x-3)^2(3x-4)^2}.$$

This is zero at $x = 17/12$. So, the coordinates of the maximum are $(17/12, 1)$.

(b) The denominator is undefined at $x = 3/2$ and $x = 4/3$, which are the vertical asymptotes. As $x \to \infty$, the fractional term tends to zero, so the horizontal asymptote is $y = 25$.

(c) With the asymptotes dotted, the graph is



(d) For $x \notin (4/3, 3/2)$, the graph is positive and slopes downwards away from the two vertical asymptotes. Hence, N-R will diverge with any starting value $x_0 \notin (4/3, 3/2)$. The iteration will converge for $x_0 \in (4/3, 3/2)$, but, if a starting point is arbitrarily chosen, it is unlikely to be in this interval.

3028. By the chain rule,

$$\frac{dy}{dx} = 5\,(10x-6)\left(5x^2-6x-1\right)^4.$$

For SPs, we need $10x - 6 = 0$ or $5x^2 - 6x - 1 = 0$. This gives $x = \frac{3}{5}, \frac{3}{5} \pm \frac{\sqrt{14}}{5}$. Since the two latter values are equidistant from the first, the three x values are in arithmetic progression.

3029. Enacting the differential operator by the product and chain rules,

$$\begin{aligned} &\frac{d}{dx}(xy^2) = 1 \\ \Longrightarrow\ & y^2 + 2xy\frac{dy}{dx} = 1 \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{1-y^2}{2xy}. \end{aligned}$$

3030. (a) Using $\sin x \equiv \cos x \tan x$,

$$\begin{aligned} &2\sin x + \tan x - 2\cos x - 1 \\ \equiv\ & 2\cos x \tan x + \tan x - 2\cos x - 1 \\ \equiv\ & (2\cos x + 1)(\tan x - 1). \end{aligned}$$

(b) For $\mathrm{h}(x) = 0$, we need $\cos x = -\frac{1}{2}$ or $\tan x = 1$. This gives $x = 45°, 120°, 225°, 240°$.

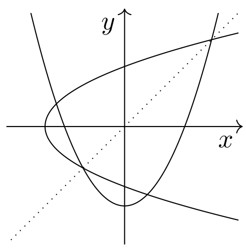
3031. (a) This is a geometric series, with first term 1 and common ratio x. So, $S_\infty = \frac{a}{1-r}$ gives

$$E_1 = \frac{1}{1-x} \equiv (1-x)^{-1} = E_2.$$

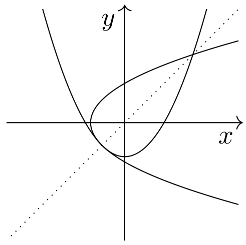
(b) The generalised binomial expansion gives

$$\begin{aligned} E_2 &= 1 + (-1)(-x) + \tfrac{(-1)(-2)}{2!}(-x)^2 + ... \\ &\equiv 1 + x + x^2 + ... \\ &= E_1. \end{aligned}$$

3032. (a) The curves are reflections in $y = x$. If k is large and negative, then there are four intersections, as shown:



The boundary case of this scenario occurs when the curves are tangent to each other. In this case, they must both have gradient -1 at the lower intersection with the line $y = x$, as shown:



(b) Solving for intersections of $y = \frac{1}{2}x^2 + k$ with $y = x$, the lower one is at $x = 1 - \sqrt{1-2k}$. The derivative of $y = \frac{1}{2}x^2 + k$ is $\frac{dy}{dx} = x$. In the boundary case, the gradient is -1 at the lower point of intersection. Hence, we need $1 - \sqrt{1-2k} = -1$. Solving this, $k = -3/2$. So, $k \in (-\infty, -3/2)$.

3033. Equating the x coordinates, $t = 3 - 2t$, so $t = 1$. At this time, the y coordinate for each particle is 2, so the particles collide at $t = 1$.

3034. We are told that $\mathrm{f}(x) - \mathrm{g}(x) = 0$ has exactly one root. Moreover, since the curves cross (as opposed to merely touch), this cannot be a double root. But a quadratic equation with exactly one root must have a double root. Hence, $\mathrm{f}(x) - \mathrm{g}(x) = 0$ cannot be a quadratic equation. So, it must be linear, which tells us that the leading coefficients of f and g are the same.

As can be seen by completing the square, two parabolae with the same leading coefficient must be translations of one another. □

3035. Looking at the directions of implication separately:

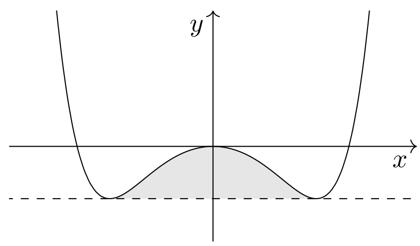
- Statement ① is $\mathrm{f}(x) = \mathrm{g}(x)(x-\alpha) + r$, where g is a polynomial. Substituting $x = \alpha$, the first term on the RHS is zero, which gives $\mathrm{f}(\alpha) = r$. So, ① $\Longrightarrow$ ②.
- We can write $\mathrm{f}(x) = \mathrm{g}(x)(x-\alpha) + R$, for some constant remainder R. Substituting $x = \alpha$ gives $\mathrm{f}(\alpha) = R$. And, since $\mathrm{f}(\alpha) = r$, we have $r = R$, which implies $\mathrm{f}(x) = \mathrm{g}(x)(x-\alpha) + r$. So, ② $\Longrightarrow$ ①.

Hence, the implication is ① $\Longleftrightarrow$ ②.

— Nota Bene —

This is known as the *remainder theorem*. It is a generalisation of the factor theorem.

3036. The scenario is as follows:



The line $y = k$ has gradient zero. So, we look for SPs. Solving $6x^5 - 2x = 0$, these are at $(0,0)$, which is not relevant, and

$$\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{2\sqrt{3}}{9}\right).$$

The curve is above the line, so the area is given by

$$A = \int_{-\frac{1}{\sqrt[4]{3}}}^{\frac{1}{\sqrt[4]{3}}} x^6 - x^2 + \tfrac{2\sqrt{3}}{9}\, dx.$$

Evaluating this with a calculator, the area enclosed is $A = 0.3342414... = 0.334$ (3sf).

3037. (a) The range of the denominator is $(0, 2a)$, so the range of the function is $(1/2a, \infty)$.
(b) The range of the denominator is $(a, 3a)$, so the range of the function is $(1/3a, 1/a)$.

3038. Using the integration facility on a calculator, the true value of the integral is

$$I_{\text{true}} = \int_0^{\frac{\pi}{6}} \sin\theta \, d\theta = 0.13397459...$$

The approximate value is

$$I_{\text{approx}} = \int_0^{\frac{\pi}{6}} 1 - \tfrac{1}{6}\theta^3 + \tfrac{1}{120}\theta^5 \, d\theta = 0.13397473...$$

So, the percentage error in the calculation is

$$E = \frac{I_{\text{approx}} - I_{\text{true}}}{I_{\text{true}}} = 0.000104\% \text{ (3sf)}.$$

3039. We substitute for y, using $b \neq 0$:

$$\begin{aligned} & b^2x^2 + a^2\left(\tfrac{-ax}{b}\right)^2 = 1 \\ \Longrightarrow\ & x^2\left(b^2 + \tfrac{a^2}{b^2}\right) = 1 \\ \Longrightarrow\ & x^2 = \frac{1}{b^2 + \frac{a^2}{b^2}}. \end{aligned}$$

The denominator $b^2 + \frac{a^2}{b^2}$ is strictly positive, so we have $x^2 = k$, where $k > 0$. This has precisely two roots $x = \pm\sqrt{k}$. And, because the first equation is linear, each of these corresponds to precisely one y value, and hence one point (x, y). □

3040. For a counterexample, we need two non-identical functions which commute, i.e. for which order of composition isn't important.

For instance, $f(x) = 2x$ and $g(x) = 3x$. The two compositions are $fg(x) = gf(x) = 6x$.

3041. We are told that $0.05 < \mathbb{P}(X \leq k)$, which means that the outcome k is not critical (it is too likely). Keeping the central statement as it is, the critical value should be $k - 1$, not k.

3042. By the chain rule,

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2-x}.$$

Setting this to zero for SPs, we need $x = 2 - x$, so $x = 1$. At $x = 1$, both $\ln x$ and $\ln(2 - x)$ are zero, so $y = 0$. Hence, the curve has a stationary point on the x axis.

— Alternative Method —

The graph is symmetrical in the line $x = 1$, since the inputs x and $2 - x$ are symmetrical in this line. Hence, $x = 1$ must be a stationary point. Evaluating at this point gives $y = 0$, so there is a stationary point on the x axis.

3043. The fan-belt exerts two forces of magnitude T on each wheel. These are symmetrical in the angle bisector of the vertex. Resolving in this direction, the magnitudes of the resultant forces, in Newtons, are

$$\begin{aligned} \text{TL} &: F_1 = 2T\cos 45°, \\ \text{BL} &: F_2 = 2T\cos 45°, \\ \text{TR} &: F_3 = 2T\cos 75°, \\ \text{BR} &: F_4 = 2T\cos 15°. \end{aligned}$$

3044. (a) Since $f(0) = -1 < 0$ and $f(1) = \frac{2}{3} > 0$, and the function has no discontinuities, there must be a root in $(0, 1)$.
(b) Differentiating, $f'(x) = 2x + \ln 3 \cdot 3^{-x}$. For $x > 0$, both terms are positive. Hence, f is an increasing function everywhere on its domain. Since it has no discontinuities, it can therefore have at most one root.

3045. The possibility space for X and Z has 36 outcomes, of which 5 give a sum of 8. So,

$$\mathbb{P}(X + Z = 8) = \tfrac{5}{36}.$$

If we are told that $X + Y = 8$, then, with equal probability, (X, Y) is one of

$$\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}.$$

Hence, $X = 2, 3, 4, 5, 6$ with equal probability. This eliminates six of the unsuccessful outcomes (those with $X = 1$) from the (X, Z) possibility space. Therefore, the probability increases to $\frac{5}{30}$.

3046. (a) The algebra is symmetrical in x and y. So, switching x and y has no effect on the graph. This means the graph must be symmetrical in the line $y = x$.

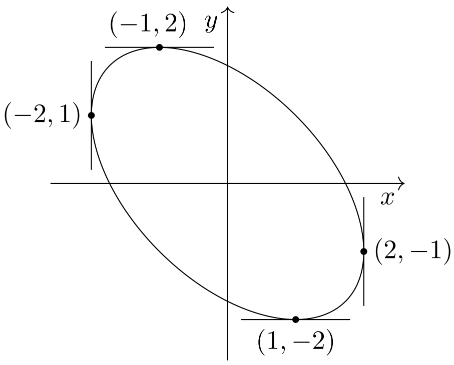
(b) Differentiating implicitly with respect to x,

$$\begin{aligned} & x^2 + xy + y^2 = 3 \\ \Longrightarrow\ & 2x + y + x\tfrac{dy}{dx} + 2y\tfrac{dy}{dx} = 0. \end{aligned}$$

Setting $\frac{dy}{dx} = 0$ gives $2x + y = 0$. Substituting this into the equation of the ellipse,

$$\begin{aligned} & x^2 + x(-2x) + (-2x)^2 = 3 \\ \Longrightarrow\ & x = \pm 1. \end{aligned}$$

So, there are tangents parallel to the x axis at $(\pm 1, \mp 2)$. And the equation $x^2 + xy + y^2 = 3$ is symmetrical in x and y, so there must be tangents parallel to the y axis at $(\pm 2, \mp 1)$.



3047. The largest possible domain of $x \mapsto \ln x$ is $(0, \infty)$. So, we require $x^2 - 1 > 0$. This has solution set $(-\infty, -1) \cup (1, \infty)$.

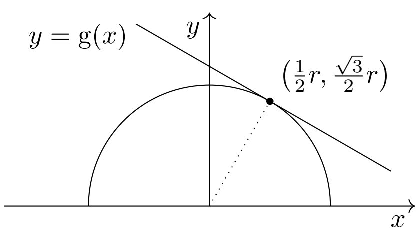
3048. (a) The resultant force on the mouse is $\frac{1}{2}mg$, so its acceleration is $\frac{1}{2}g$. Starting from $u = 0$, this gives $h = \frac{1}{4}gt^2$ and $v = \frac{1}{2}gt$.

(b) Once the mouse has let go, its acceleration is $-g$. Its initial velocity is $u = \frac{1}{2}gt_0$, and it has a displacement of $s = -\frac{1}{4}gt_0^2$ to fall. This gives

$$\begin{aligned} & v^2 = \left(\tfrac{1}{2}gt_0\right)^2 + 2 \cdot -g\left(-\tfrac{1}{4}gt_0^2\right) \\ \Longrightarrow\ & v^2 = \tfrac{3}{4}g^2t_0^2. \end{aligned}$$

So, landing speed is $\frac{\sqrt{3}}{2}gt_0$ ms^{-1}.

3049. Consider $y = \mathrm{f}(x)$. This is a semicircle, radius r, centred on O. Also consider $y = \mathrm{g}(x)$. This is the equation of a straight line. Since the values of the functions f and g, and also their derivatives, match at $x = \frac{r}{2}$, the line $y = \mathrm{g}(x)$ must be tangent to the semicircle at this point:

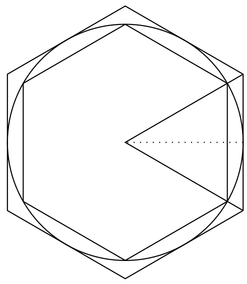


The radius at this point has gradient $\sqrt{3}$, so the tangent has gradient $-\frac{1}{\sqrt{3}}$. This gives

$$\begin{aligned} & y - \tfrac{\sqrt{3}}{2}r = -\tfrac{1}{\sqrt{3}}(x - \tfrac{1}{2}r) \\ \Longrightarrow\ & y = \tfrac{1}{\sqrt{3}}(2r - x). \end{aligned}$$

So, the required function is $\mathrm{g} : x \mapsto \frac{1}{\sqrt{3}}(2r - x)$.

3050. With a circle marking the path of the vertices of the inner hexagon, the scenario is:



By trigonometry, the ratio of the radii is $2 : \sqrt{3}$. Hence, the ratio of areas is $4 : 3$. So, the largest inner hexagon has area 3.

3051. Multiplying up by the denominators,

$$(1 - \sqrt{x})^3 + (1 + \sqrt{x})^3 = (1 - \sqrt{x})^3(1 + \sqrt{x})^3.$$

The RHS simplifies to $(1 - x)^3$. We can then use three binomial expansions. Half of the terms on the LHS cancel, leaving

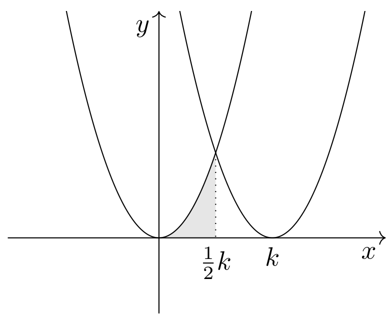
$$\begin{aligned} & 2 + 6x = 1 - 3x + 3x^2 - x^3 \\ \Longrightarrow\ & x^3 - 3x^2 + 9x + 1 = 0. \end{aligned}$$

This is a cubic equation, so must have at least one real root. To prove it has no others, we find

$$\frac{d}{dx}(x^3 - 3x^2 + 9x + 1) \equiv 3x^2 - 6x + 9.$$

This has discriminant $\Delta = -72 < 0$, so the cubic is increasing everywhere. Hence, it has exactly one real root, as required.

3052. The parabolae have points of tangency on the x axis at $x = 0$ and $x = k$ respectively:



Since the graphs are symmetrical, we find the area shaded above, and double it. Again by symmetry, the curves intersect at $x = \frac{1}{2}k$:

$$\begin{aligned} A &= 2\int_0^{\frac{1}{2}k} x^2\,dx \\ &\equiv 2\Big[\tfrac{1}{3}x^3\Big]_0^{\frac{1}{2}k} \\ &\equiv \tfrac{1}{12}k^3, \text{ as required.} \end{aligned}$$

3053. (a) There is weak positive correlation between the variables I_1 and I_2.

(b) There are significantly fewer people who made losses in the first year, compared to those who made losses in the second year.

(c) The sample is taken from those registered as self-employed for both years. Not included are those who were self-employed for the first year, but not for the second. These people are much more likely to have been those who made losses in the first year. Hence, the sampling method is biased towards net profits in the first year.

3054. Let $x = \sin\theta$. Then $dx = \cos\theta\,d\theta$. The x limits were -1 to 1, so the u limits are $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The area required is twice the area of the semicircle:

$$\begin{aligned} A &= 2\int_{-1}^{1} \sqrt{1-x^2}\,dx \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta. \end{aligned}$$

Using a double-angle formula, this is

$$\begin{aligned} &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta + 1\,d\theta \\ &= \Big[\tfrac{1}{2}\sin 2\theta + \theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \pi, \text{ as required.} \end{aligned}$$

— Nota Bene —

The sine and cosine functions, and the radians in which their angles are measured, are themselves defined with reference to the unit circle. So, the above could be seen as more of a verification than a proof.

3055. We are told that

$$\frac{d}{dx}\left(\frac{y^2}{x+y}\right) = 0.$$

Using the quotient rule,

$$\frac{2y\frac{dy}{dx}(x+y) - y^2\left(1+\frac{dy}{dx}\right)}{(x+y)^2} = 0.$$

Setting the numerator to zero,

$$\begin{aligned} (2xy+2y^2)\tfrac{dy}{dx} - y^2 - y^2\tfrac{dy}{dx} &= 0 \\ \Longrightarrow \frac{dy}{dx} &= \frac{y^2}{2xy+y^2} \\ &= \frac{y}{2x+y}, \text{ as required.} \end{aligned}$$

— Alternative Method —

We are told that

$$\frac{d}{dx}\left(\frac{y^2}{x+y}\right) = 0.$$

Integrating both sides with respect to x,

$$\begin{aligned} \frac{y^2}{x+y} &= k \\ \Longrightarrow y^2 &= kx + ky. \end{aligned}$$

Differentiating with respect to x,

$$\begin{aligned} 2y\tfrac{dy}{dx} &= k + k\tfrac{dy}{dx} \\ \Longrightarrow \tfrac{dy}{dx} &= \frac{k}{2y-k} \\ &= \frac{\frac{y^2}{x+y}}{2y - \frac{y^2}{x+y}} \\ &= \frac{y}{2x+y}, \text{ as required.} \end{aligned}$$

3056. Statement ② is false. The relevant double-angle formula is $\sin 2\phi \equiv 2\sin\phi\cos\phi$. Hence, $\phi = \frac{\pi}{2}$, for which $\cos\phi = 0$, is a counterexample.

3057. (a) The derivative of e^{-2x} is $-2e^{-2x}$. Hence

$$\frac{dy}{dx} + 2y = -2e^{-2x} + 2e^{-2x} = 0.$$

So $y = e^{-2x}$ satisfies the DE.

(b) i. Differentiating by the product rule,

$$\frac{dy}{dx} = \mathrm{f}'(x)e^{-2x} - 2\,\mathrm{f}(x)e^{-2x}.$$

ii. Substituting into the DE,

$$\begin{aligned} \mathrm{f}'(x)e^{-2x} - 2\,\mathrm{f}(x)e^{-2x} + 2\,\mathrm{f}(x)e^{-2x} &= 0 \\ \Longrightarrow \mathrm{f}'(x)e^{-2x} &= 0. \end{aligned}$$

Since $e^{-2x} \neq 0$, this gives $\mathrm{f}'(x) = 0$.

iii. Integrating the above, $\mathrm{f}(x) = A$ for some constant A. So, all solution curves must be of the form $y = Ae^{-2x}$.

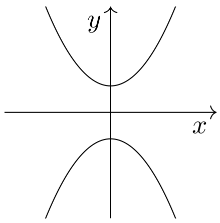
3058. This is a quadratic in $\sin t$:

$$\begin{aligned} &2\operatorname{cosec} t + \sin t = 3 \\ \Longrightarrow\ &\sin^2 t - 3\sin t + 2 = 0 \\ \Longrightarrow\ &(\sin t - 1)(\sin t - 2) = 0 \\ \Longrightarrow\ &\sin t = 1, 2. \end{aligned}$$

The latter gives no roots, since the range of $\sin t$ is $[-1, 1]$. Hence, $t = \frac{\pi}{2}$.

3059. (a) Yes. An even polynomial function has the y axis as a line of symmetry. So, the curves $y = \mathrm{f}(x)$ and $y = \mathrm{f}(-x)$ are identical. They intersect at all x values.

(b) No. $\mathrm{f}(x) = x^2 + 1$ is a counterexample. The curves $y = x^2 + 1$ and $y = -x^2 - 1$ do not intersect, as shown below:



(c) No. $\mathrm{f}(x) = x^2 + 1$ is still a counterexample.

3060. Splitting into partial fractions,

$$\frac{2}{1-x^2} = \frac{1}{1+x} + \frac{1}{1-x},$$
$$\frac{2}{4-x^2} = \frac{1}{2}\left(\frac{1}{2+x} + \frac{1}{2-x}\right).$$

We can now integrate. In each case, the second partial fraction generates a negative sign by the reverse chain rule, which then produces division via a log rule:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2}{1-x^2} + \frac{2}{4-x^2}\,dx$$
$$= \left[\ln\left|\tfrac{1+x}{1-x}\right| + \tfrac{1}{2}\ln\left|\tfrac{2+x}{2-x}\right|\right]_{-\frac{1}{2}}^{\frac{1}{2}}.$$

Simplifying with various log rules, this is

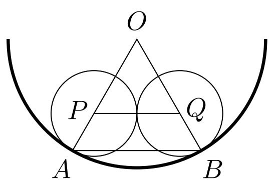
$$\begin{aligned} &\left(\ln 3 + \tfrac{1}{2}\ln\tfrac{5}{3}\right) - \left(\ln\tfrac{1}{3} + \tfrac{1}{2}\ln\tfrac{3}{5}\right) \\ &= 2\ln 3 + \ln\tfrac{5}{3} \\ &= \ln\left(3^2 \cdot \tfrac{5}{3}\right) \\ &= \ln 15, \text{ as required.} \end{aligned}$$

3061. (a) $x \neq y \iff \sin x \neq \sin y$. A counterexample to the implication in the opposite direction is $x = 0$, $y = \pi$.

(b) $x \neq y \iff \arcsin x \neq \arcsin y$, because arcsin is a one-to-one function.

(c) $x \neq y \impliedby |x| \neq |y|$. A counterexample to the implication in the opposite direction is $x = -1$, $y = 1$.

3062. Drawing in the various radii etc.,



The diameter of the spheres is $2r$. So, the radii OA and OB, of length $3r$, are trisected. Hence, $\triangle OPQ$ is equilateral, with side length $2r$. And $\triangle OAB$ is similar to $\triangle OPQ$, so $|AB| = 3r$.

3063. Using the quotient rule, the first derivative is

$$\frac{dy}{dx} = \frac{k(k-x^2)}{(k+x^2)^2}.$$

The second derivative is

$$\frac{d^2y}{dx^2} = \frac{2kx(x^2-3k)}{(k+x^2)^3}.$$

Since $k > 0$, the denominator is never zero, so the second derivative is well defined everywhere. The numerator factorises as

$$2kx\left(x - \sqrt{3k}\right)\left(x + \sqrt{3k}\right).$$

This has roots at $x = 0, \pm\sqrt{3k}$. Since these are single roots, the second derivative changes sign at each. Hence, there are points of inflection at

$$\left(-\sqrt{3k}, -\tfrac{1}{4}\sqrt{3k}\right) \quad (0,0) \quad \left(\sqrt{3k}, \tfrac{1}{4}\sqrt{3k}\right).$$

These points have rotational symmetry around the origin, so they are collinear, as required.

3064. Since $\mathrm{f}(x)$ is quadratic in $e^x \cos x$, we can find the range by completing the square:

$$\mathrm{f}(x) = \left(e^x \cos x + \tfrac{1}{2}\right)^2 - \tfrac{1}{4}.$$

The range of the function $x \mapsto e^x \cos x$ is $\mathbb{R}$, so the range of f is $[-1/4, \infty)$.

3065. Let $y = \mathrm{f}(x)$ and $z = \mathrm{g}(x)$. Then the chain rule is

$$\begin{aligned} \frac{dy}{dz} &\equiv \frac{dy}{dx} \times \frac{dx}{dz} \\ &\equiv \frac{dy}{dx} \div \frac{dz}{dx} \\ &= \mathrm{f}'(x) \div \mathrm{g}'(x). \end{aligned}$$

Hence, $\dfrac{d(\mathrm{f}(x))}{d(\mathrm{g}(x))} \equiv \dfrac{\mathrm{f}'(x)}{\mathrm{g}'(x)}$, as required.

3066. A stretch scale factor k in the y direction gives $y = k(x-a)^2 + bk$. Then translating by vector $(k - bk)\mathbf{j}$ gives $y = k(x-a)^2 + b$.

— Nota Bene —

This combination of transformations can also be expressed as a single enlargement, scale factor k, with invariant line $y = b$.

Referring to an invariant line is a compact way of describing a 1D stretch.

- A stretch parallel to the x axis may be put as "in the x direction" or "with $y = 0$ invariant".

- A stretch parallel to the y axis may be put as "in the y direction" or "with $x = 0$ invariant".

When one of the *axes* is invariant, as above, then these are as succinct as each other. But a stretch by scale factor 2 with $x = 3$ invariant would have to be described as a stretch by scale factor 2 in the x direction followed by a translation by vector $-3\mathbf{j}$. In such an example, the language of invariant lines is more effective.

3067. This is false. The function $f(x) = x^4 + x$, at $x = 0$, is a counterexample: $f'''(0) = 0$, but $f'(0) = 1$.

3068. We can rewrite as

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx.$$

The top is the derivative of the bottom. It is a standard result that

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c.$$

This gives

$$\begin{aligned}\int \tan x \, dx &= -\ln|\cos x| + c \\ &\equiv \ln|(\cos x)^{-1}| + c \\ &\equiv \ln|\sec x| + c, \text{ as required.}\end{aligned}$$

——— Alternative Method ———

Using the chain rule, if $\sec x \geq 0$, then,

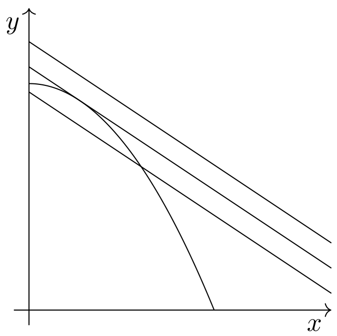
$$\frac{d}{dx}\big(\ln(\sec x)\big) = \frac{1}{\sec x} \cdot \tan x \sec x = \tan x.$$

Or, if $\sec x < 0$, then,

$$\frac{d}{dx}\big(\ln(-\sec x)\big) = \frac{1}{-\sec x} \cdot -\tan x \sec x = \tan x.$$

This proves the integral result.

3069. (a) Below is the curve $y = 3 - \frac{1}{2}x^2$ and three profit lines:

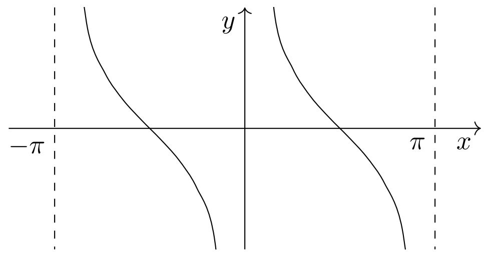


Increasing P moves the profit line up and to the right. When it has been moved as far as possible, while still remaining in contact with the parabola, it will be a tangent.

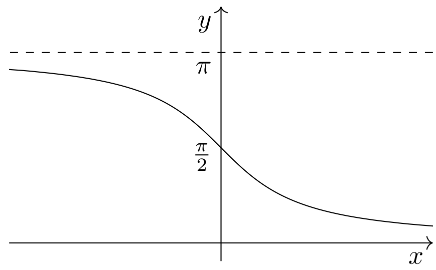
(b) The gradient of the parabola is $\frac{dy}{dx} = -x$, so we set $-x = -\frac{2}{3}$, which gives $x = \frac{2}{3}, y = \frac{25}{9}$, so $P_{\max} = 29$.

3070. There are 8 ways of placing a pawn in the first row. Once this has been done, there are 7 ways of placing a pawn in the second row, and so on. This gives $8! = 40320$ ways.

3071. Since $\cot x \equiv \tan\left(\frac{\pi}{2} - x\right)$, the graph $y = \cot x$ is a reflection of $y = \tan x$ in the line $x = \frac{\pi}{4}$:



To render this invertible, we restrict the domain to $(0, \pi)$. The codomain is $\mathbb{R}$. Inverting, the domain of the cot function is $\mathbb{R}$ and the codomain is $(0, \pi)$. Graphically, we reflect the right-hand branch of the above in the line $y = x$:



3072. Multiplying up by the denominator, we raise both sides to the power of 6:

$$\begin{aligned}\frac{\sqrt{x+1}}{\sqrt[3]{x-1}} &= 1 \\ \Longrightarrow \sqrt{x+1} &= \sqrt[3]{x-1} \\ \Longrightarrow (x+1)^3 &= (x-1)^2 \\ \Longrightarrow x^3 + 2x^2 + 5x &= 0.\end{aligned}$$

The discriminant of the quadratic factor in the above is $\Delta = -16 < 0$. So, the only possible root is $x = 0$. However, testing $x = 0$ in the original equation, the LHS is $1/-1 \neq 1$. So, the equation has no real roots.

——— Nota Bene ———

The phantom root $x = 0$ has been introduced by raising both sides to the power 6, which is *even*.

3073. The rectangle has unspecified length, so we need only show that one of the perpendicular heights (altitudes) of the triangle is shorter than 6 cm. The shortest altitude is perpendicular to the longest side. Calculating its length l cm using the sine and cosine rules,

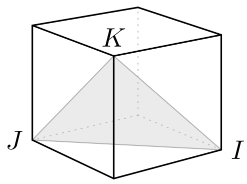
$$l = \tfrac{8\sqrt{5}}{3} \approx 5.96 < 6.$$

Hence, with the 9 cm edge laid parallel to one side, the triangle will fit inside the rectangle.

3074. (a) A cubic is rotationally symmetrical about its point of inflection. If this is at the origin, then the (monic) curve must have the form $y = x^3 + kx$, for some k. Then $\frac{dy}{dx} = 3x^2 + k$. The origin is a stationary point iff $k = 0$. So, the curve in question must be a translation of $y = x^3$, of the form $y = (x - p)^3 + q$.

(b) According to the argument above, $p = -5$ and $q = 6$.

3075. The triangle is rotationally symmetrical (about the line $x = y = z$), so it is equilateral. In the diagram, I is the point with position vector $\mathbf{i}$, and so on:



It has side length $\sqrt{2}$, so its area is

$$A = \tfrac{1}{2}\big(\sqrt{2}\big)^2 \sin 60° = \tfrac{\sqrt{3}}{2}.$$

3076. The original possibility space is ${}^{10}\text{C}_3 = 120$ sets of three bottles that could fall. The condition given restricts this to ${}^{8}\text{C}_3 = 56$ sets. For success, the fallen bottles form one contiguous group ABC. This group could have $\{1, 2, 3, 4, 5, 6\}$ unfallen bottles to its left. So, the probability is $\frac{6}{56} = \frac{3}{28}$.

3077. Horizontally, $x = u_x t$. Vertically, $y = 0.25t - 5t^2$. Substituting the former into the latter,

$$y = 0.25\frac{x}{u_x} - 5\frac{x^2}{u_x^2}.$$

Multiplying by 5 and simplifying, this is

$$5y = \frac{5}{4u_x}x - \frac{25}{u_x^2}x^2.$$

Equating coefficients, $u_x = 0.15625$ and $k = 1024$.

3078. Substituting for $x + y$, we get $2xy = 2$, so $xy = 1$. Substituting this back into $x + y = 2$, we get $x + \frac{1}{x} = 2$. Multiplying by x, this simplifies to $(x - 1)^2 = 0$. Since $x = 1$ is a double root, the point of intersection at $(1, 1)$ is a point of tangency.

— Alternative Method —

Differentiating implicitly,

$$\begin{aligned} &x^2y + xy^2 = 2 \\ \Longrightarrow\ &2xy + x^2\tfrac{dy}{dx} + y^2 + 2xy\tfrac{dy}{dx} = 0. \end{aligned}$$

Setting $\frac{dy}{dx} = -1$,

$$\begin{aligned} &2xy - x^2 + y^2 - 2xy = 0 \\ \Longrightarrow\ &y = \pm x. \end{aligned}$$

Substituting this back to the equation of the curve,

$$\begin{aligned} &x^2(\pm x) + x^3 = 2 \\ \Longrightarrow\ &x = 1. \end{aligned}$$

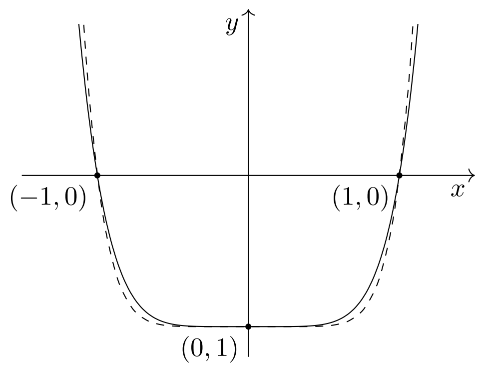
So, the gradient is -1 at $(1, 1)$. This also satisfies $x + y = 2$. Hence, the line $x + y = 2$ is a tangent to the curve.

3079. Writing $\cot x \equiv \frac{\cos x}{\sin x}$, we use the quotient rule:

$$\begin{aligned} &\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\ \equiv\ &\frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} \\ \equiv\ &-(\sin^2 x + \cos^2 x)\operatorname{cosec}^2 x \\ \equiv\ &-\operatorname{cosec}^2 x, \text{ as required.} \end{aligned}$$

3080. In a pack with the aces removed, there are 12 spades and 12 face cards. So, the probabilities of these two events are equal.

3081. The graphs $y = x^6$ and $y = x^8$ are positive even polynomials which intersect at $(0, 0)$ and $(\pm 1, 1)$. Outside this domain, $y = x^8$ lies above x^6. The -1's then translate this picture by vector $-\mathbf{j}$. The dashed line is the octic curve:

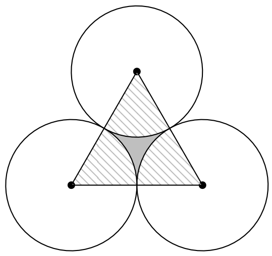


3082. Multiplying up by the denominators of the inlaid fractions,

$$\begin{aligned}\frac{1+\frac{1}{x}}{1-\frac{1}{x}} &= \frac{2+\frac{1}{x^2}}{1-\frac{1}{x^2}}\\ \Longrightarrow \frac{x+1}{x-1} &= \frac{2x^2+1}{x^2-1}\\ \Longrightarrow (x+1)^2 &= 2x^2+1\\ \Longrightarrow x &= 0, 2.\end{aligned}$$

We reject $x = 0$, for which the original fractions are undefined, leaving $x = 2$.

3083. The equilateral triangle formed by the centres has side length 2, and therefore area $\frac{1}{2}2^2 \sin 60 = \sqrt{3}$.



The three sectors (hatched) collectively form the interior of a semicircle of radius 1, with area $\pi/2$. Hence, the area of the shaded region is $\sqrt{3} - \pi/2$, as required.

3084. In the form $\text{f}(x) = \text{g}(y)$, the graph is $x^2 - x = -y^3$. The left-hand side is a quadratic, on which we can complete the square to give $\left(x - \frac{1}{2}\right)^2 = \frac{1}{4} - y^3$. The nature of the RHS isn't relevant: the equation is symmetrical around the value $x = \frac{1}{2}$, therefore the graph has a line of symmetry at $x = \frac{1}{2}$.

3085. According to the coefficient of friction model, there is a value for friction F_{max} which doesn't depend on the relative speed of the surfaces. Hence, if the tablecloth is pulled away very quickly, this doesn't increase the force acting on the glasses. Therefore, the acceleration of the glasses is, according to the coefficient of friction model, independent of cloth speed. Call it a_0. The total sideways speed gained by the glasses is then $v = a_0 \Delta t$, where Δt is the time for which the moving cloth is in contact with the glasses. By reducing Δt sufficiently, v can be reduced to a speed which will not knock the glasses over.

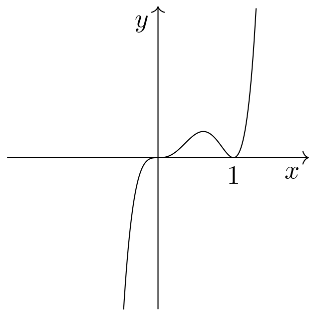
3086. Differentiating implicitly,

$$\begin{aligned}e^y &= \frac{x^2+3}{x-1}\\ \Longrightarrow e^y \frac{dy}{dx} &= \frac{x^2-2x-3}{(x-1)^2}.\end{aligned}$$

Setting $\frac{dy}{dx} = 0$, we get $x^2 - 2x - 3 = 0$, which has solution $x = -1, 3$. The negative value is outside the domain, so the turning point has coordinates $(3, \ln 6)$.

3087. There are twelve successful outcomes, eight with common difference ± 1 and lowest value $\{1, 2, 3, 4\}$, and four with common difference ± 2 and lowest value $\{1, 2\}$. Hence, the probability is $\frac{12}{216} = \frac{1}{18}$.

3088. Factorising, we have $y = x^3(x-1)^2$. This is a positive quintic with a triple root at $x = 0$ and a double root at $x = 1$. So, the graph is



3089. Using $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, for small $|x|$,

$$\begin{aligned}\text{h}(x) &= (x+1)\cos\left(x^2\right)\\ &\approx (x+1)\left(1 - \tfrac{1}{2}\left(x^2\right)^2\right)\\ &\equiv 1 + x - \tfrac{1}{2}x^4 - \tfrac{1}{2}x^5.\end{aligned}$$

Since the terms in x^2 and x^3 are both zero, the function h is very well approximated by first two terms, i.e. by the linear function $x \mapsto 1 + x$.

3090. Multiplying the variance by n^2 gives a quadratic:

$$n^2 s^2 = n \sum x^2 - \left(\sum x\right)^2.$$

Substituting values,

$$\begin{aligned}1.1476n^2 &= 2275n - 705^2\\ \Longrightarrow \quad n &= 250, 1732.4...\end{aligned}$$

We reject the non-integer value, so $n = 250$.

3091. Substituting the formulae for x and y into the double-angle identity $\cos 2t \equiv 1 - 2\sin^2 t$, we get a parabola: $x = 1 - \frac{1}{2}y^2$.

3092. Each equation offers two options, at least one of which must hold:

$$\begin{aligned}(x-a)(y-b) = 0 &\Longrightarrow x = a \text{ and/or } y = b,\\ (y-b)(z-c) = 0 &\Longrightarrow y = b \text{ and/or } z = c,\\ (z-c)(x-a) = 0 &\Longrightarrow z = c \text{ and/or } x = a.\end{aligned}$$

These are satisfied if at least two of (x, y, z) take the respective values (a, b, c). The last variable can take any value. So, (x, y, z) is (a, b, k_1), (a, k_2, c), or (k_3, b, c), where $k_1, k_2, k_3 \in \mathbb{R}$.

3093. Solving simultaneously, there are intersections at (a, a^2) and, if $a \neq 0$, another point. At (a, a^2), the curve has gradient $2a$. The line has gradient $-\frac{1}{2a}$. These are negative reciprocals, so the line and the parabola intersect at right angles.

— Nota Bene —

The other intersection is not right-angled. But this doesn't interfere with the result required by the question.

3094. The resultant force is 8 N. Since $8 > 4+5+6$, there are infinitely many ways of adding three vectors of magnitudes $4, 5, 6$ to produce a vector of length 8. For an explicit counterexample, we could construct vector-sum triangles with lengths

$(9, 6, 8)$, with the 4 and 5 N forces parallel,
$(10, 5, 8)$, with the 4 and 6 N forces parallel.

3095. The successful triangles are equilateral, with each side a face diagonal of length $\sqrt{2}$. Choose a vertex without loss of generality. The probability that the next vertex chosen lies along a face diagonal is $3/7$. There are now 6 vertices to choose from, of which 2 give successful triangles. So, the probability is $3/7 \times 2/6 = 1/7$.

— Alternative Method —

The possibility space consists of the ${}^8C_3 = 56$ sets of vertices which can be chosen. Consider a face $ABCD$ of the cube. Classify successful triangles by the number of vertices on face $ABCD$:

① One vertex on $ABCD$. There are four to choose from. The other two vertices are on the opposite face, offering one possibility. This gives 4 successful outcomes.

② Two vertices on $ABCD$. These can be A, C or B, D. Each offers two possibilities for the third vertex. This gives $2 \times 2 = 4$ successful outcomes.

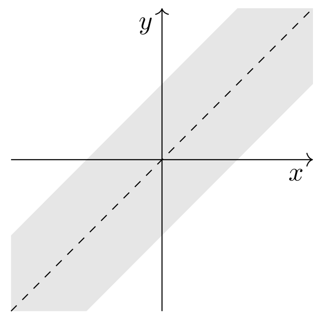
Hence, $p = \frac{\text{successful}}{\text{total}} = \frac{8}{56} = \frac{1}{7}$.

3096. Every derivative of e^x is e^x. So, at $x = 0$, every derivative has value 1.

(a) The first two derivatives of the approximation are $1 + x$ and 1. At $x = 0$, both have the correct value 1.

(b) The third derivative of the approximation has value 0, which is incorrect.

3097. The circles have radius $\sqrt{2}$, and their centres are at $(-k, k)$. The centres generate the line $y = -x$. The region R consists of all points lying within $\sqrt{2}$ of $y = -x$. The boundary lines of R intersect the axes at ± 1.



3098. (a) The function is undefined when $1 - \sin x = 0$, which is $x = \frac{\pi}{2} + 2n\pi$, for $n \in \mathbb{Z}$. So, D is $\mathbb{R} \setminus \{x : x = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}\}$.

(b) Writing $\text{f}(x)$ as a proper fraction,

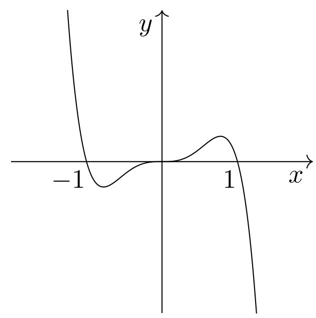
$$\text{f} : x \mapsto \frac{\sin x}{1 - \sin x} \equiv -1 + \frac{1}{1 - \sin x}.$$

Since $1 - \sin x$ can take any value in $[0, 2]$, its reciprocal can take any value in $[1/2, \infty)$. Subtracting 1, the range of the function f is $[-1/2, \infty)$.

3099. The t limits are given by $1 + \frac{1}{t} = 2, 3$, so $t_1 = 1$ and $t_2 = \frac{1}{2}$. Using the parametric integration formula, the area of the region is

$$\begin{aligned}\int_{t_1}^{t_2} y \frac{dx}{dt}\, dt &= \int_1^{\frac{1}{2}} t^2 \cdot -t^{-2}\, dt \\ &= \int_1^{\frac{1}{2}} -1\, dt \\ &= \Big[-t\Big]_1^{\frac{1}{2}} \\ &= \tfrac{1}{2}.\end{aligned}$$

3100. Factorising, we have $y = x^3(1 - x)(1 + x)$. So, the curve is a negative quintic with a triple root (stationary point of inflection) at $x = 0$ and single roots at $x = \pm 1$:



— End of 31st Hundred —